B.Sc Part I
Paper I
Hyperbolic Functions (Examples)

Ex-1: If $u = \log_e \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$, Prove that
(i) $\tanh\frac{u}{2} = \tan\frac{x}{2}$ (ii) $x = -i\log\tan\left(\frac{iu}{2}+\frac{\pi}{4}\right)$

Ans:— Given that $u = \log_e \tan\left(\frac{\pi}{4}+x/2\right)$

$$\Rightarrow e^u = \tan\frac{\pi}{4}+\tan x/2 \Big/ 1-\tan\frac{\pi}{4}.\tan x/2$$

$$\Rightarrow e^{\frac{u}{2}+\frac{u}{2}} = \frac{1+\tan\frac{x}{2}}{1-\tan x/2} \Rightarrow \frac{e^{u/2}}{e^{-u/2}} = \frac{1+\tan x/2}{1-\tan x/2}$$

Apply Componendo and dividendo, we get

$$\frac{e^{u/2}+e^{-u/2}}{e^{u/2}-e^{-u/2}} = \frac{1+\tan x/2+1-\tan x/2}{\cancel{1}+\tan x/2-\cancel{1}+\tan x/2}$$

$$\Rightarrow \frac{2\cosh\frac{u}{2}}{2\sinh u/2} = \frac{2}{2\tan x/2}$$

$$\Rightarrow \coth\frac{u}{2} = \frac{1}{\tan x/2} \Rightarrow \tan x/2 = \tanh\frac{u}{2}$$

$$\therefore \tanh\frac{u}{2} = \tan\frac{x}{2} \text{ Proved.}$$

(ii) we have $\tan x/2 = \tanh u/2$

$$\Rightarrow \frac{i\sin\frac{x}{2}}{\cos x/2} = i\left(\frac{\tan i\frac{u}{2}}{i}\right) = \tan\frac{iu}{2}$$

Apply componendo and dividendo, we get

$$\frac{\cos\frac{x}{2}+i\sin\frac{x}{2}}{\cos\frac{x}{2}-i\sin x/2} = \frac{1+\tan\frac{iu}{2}}{1-\tan\frac{iu}{2}}$$

$$\frac{e^{i\frac{x}{2}}}{e^{-ix/2}} = \frac{\tan\frac{\pi}{4}+\tan\frac{iu}{2}}{1-\tan\frac{\pi}{4}\tan\frac{iu}{2}}$$

$$\Rightarrow e^{i\frac{x}{2}+i\frac{x}{2}} = \tan\left(\frac{\pi}{4}+\frac{iu}{2}\right)$$

$$\Rightarrow e^{ix} = \tan\left(\frac{\pi}{4}+\frac{iu}{2}\right)$$

Taking logarithms both side, we get

$$\log e^{ix} = \log \tan\left(\frac{\pi}{4}+\frac{iy}{2}\right)$$

$$\Rightarrow ix \log e = \log \tan\left(\frac{\pi}{4}+\frac{iy}{2}\right)$$

$$\Rightarrow x = -i \log \tan\left(\frac{\pi}{4}+\frac{iy}{2}\right)$$

Proved

Ex – Separate into real and imaginary parts the expressions

(a) $\cot(\alpha+i\beta)$ (b) $\tanh(\alpha+i\beta)$ (c) $\tan^{-1}(\alpha+i\beta)$.
(d) $\sinh(x+iy)$ (e) $\cosh(x+iy)$.

Ans: – (a) We have $\cot(\alpha+i\beta) = \frac{\cos(\alpha+i\beta)}{\sin(\alpha+i\beta)}$

$$\Rightarrow \cos(\alpha+i\beta) = \frac{2\cos(\alpha+i\beta)\sin(\alpha-i\beta)}{2\sin(\alpha+i\beta)\sin(\alpha-i\beta)}$$

$$\Rightarrow \cot(\alpha+i\beta) = \frac{\sin 2\alpha - \sin 2i\beta}{\cos 2i\beta - \cos 2\alpha}$$

$$\cot(\alpha+i\beta) = \frac{\sin 2\alpha - i\sinh 2\beta}{\cosh 2\beta - \cos 2\alpha}$$

$$\cot(\alpha+i\beta) = \left(\frac{\sin 2\alpha}{\cosh 2\beta - \cos 2\alpha}\right) + i\left(\frac{-\sinh 2\beta}{\cosh 2\beta - \cos 2\alpha}\right)$$

Real + imaginary.

(b) $\tanh(\alpha+i\beta) = \frac{\sinh(\alpha+i\beta)}{\cosh(\alpha+i\beta)} = \frac{\sin i(\alpha+i\beta)}{i\cos i(\alpha+i\beta)}$

$$\Rightarrow \tanh(\alpha+i\beta) = \frac{i\sin(i\alpha-\beta)}{i\cos(i\alpha-\beta)} = \frac{i\,2\sin(i\alpha-\beta)\cos(i\alpha-\beta)}{i\,2\cos(i\alpha-\beta)\cos(i\alpha+\beta)}$$

$$= \frac{i\sin 2i\alpha - \sin 2\beta}{i\cos 2i\alpha + \cos 2\beta} = \frac{1}{i}\left[\frac{i\sinh 2\alpha - \sin 2\beta}{\cosh 2\alpha + \cos 2\beta}\right]$$

$$\tanh(\alpha+i\beta) = \frac{\sinh 2\alpha}{\cosh 2\alpha + \cos 2\beta} + i\,\frac{\sin 2\beta}{\cosh 2\alpha + \cos 2\beta}$$

= Real + Imaginary.

(c) Let $\tan^{-1}(\alpha+i\beta) = x+iy \Rightarrow \tan(x+iy) = \alpha+i\beta$.

Replace i by $-i$, we get

$$\tan(x-iy) = \alpha - i\beta$$

We have $\tan 2x = \tan(x+x+iy-iy)$

$= \tan\{(x+iy)+(x-iy)\}$

$= \dfrac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\cdot\tan(x-iy)}$

$= \dfrac{(\alpha+i\beta)+(\alpha-i\beta)}{1-(\alpha+i\beta)(\alpha-i\beta)}$

$\tan 2x = \dfrac{2\alpha}{1-(\alpha^2-i^2\beta^2)} = 2\alpha/1-\alpha^2-\beta^2$

$\Rightarrow 2x = \tan^{-1}\left[2\alpha/(1-\alpha^2-\beta^2\right]$

$\Rightarrow x = \frac{1}{2}\tan^{-1}\left[\frac{2\alpha}{1-\alpha^2-\beta^2}\right]$

In the Same Way

$\tan(2iy) = \tan\{(x+iy)-(x-iy)\}$.

$\tan(2iy) = \dfrac{2i\beta}{1+\alpha^2+\beta^2}$

$\Rightarrow i\tanh 2y = \dfrac{2i\beta}{1+\alpha^2+\beta^2}$ $[\because \tan iy = i\tanh y]$

$\Rightarrow 2y = \tanh^{-1}\dfrac{2\beta}{1+\alpha^2-\beta^2}$

Using (2) and (3) in (1), we get

$\tan^{-1}(\alpha+i\beta) = x+iy$

$\tan^{-1}(\alpha+i\beta) = \frac{1}{2}\tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right) + i\frac{1}{2}\tanh^{-1}\frac{2\beta}{1+\alpha^2+\beta^2}$.

Real

(d) We have

$\sinh(x+iy) = \frac{1}{i}\sin\{i(x+iy)\} = \frac{i}{i^2}\sin(ix+i^2y)$

$= -i\sin(ix-y) = -i(\sin ix\cos y - \cos ix\cdot\sin y)$

$= -i\sin ix\cos y + i\cos ix\sin y$

$\sinh(x+iy) = \sinh x\cos y + i\cosh x\sin y$

$=$ Real + Imaginary.

(e) We have $\cosh(x+iy) = \cosh x\cosh iy + \sinh x\sinh iy$

$= \cosh x\cos i(iy) + \sinh x\frac{1}{i}\sin i(iy)$

$= \cosh x\cos(-y) + \sinh x\cdot\frac{i}{i^2}\sin(-y)$

$\cosh(x+iy) = \cosh x\cos y + i\sinh x\sin y$

Real + Imaginary.

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